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Fibonacci Sequence and Continued Fraction Expansions in Real Quadratic Number Fields

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ABSTRACT

In 2002, Tomita and Yamamuro defined several theorems for fundamental unit of certain real quadratic number fields. Although, there are infinitely many values of d having all 1s in the symmetric part of continued fraction expansion of w_d, Tomita and Yamamuro (1992) had described explicitly one type of d for the fundamental units of the real quadratic fields by using Fibonacci sequence in the Theorem 3 for $d \equiv 2,3(mod4)$ and in the Theorem 2 in the case of $d \equiv 1(mod4)$ (2002). The main purpose of this paper is to generalize and provide an improvement of the theorem 3 and the theorem 2 in the paper of Tomita and Yamamuro (2002). Moreover, the present paper deals with new certain formulas for fundamental unit ε_d and Yokoi's d -invariants n_d, m_d in the relation to continued fraction expansion of w_d for such real quadratic fields. All results are supported by numerical tables.

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1. Introduction

Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic number field where $d > 0$ is a positive square free integer. In real quadratic fields, integral basis element is denoted by $w_d = \sqrt{d} = [\,a_0;\overline{a_1, a_2, \ldots, a_{\ell(\mathrm{d})-1}, 2a_0}\,]$ for $d \equiv 2,3(mod 4)$ and $w_d = \frac{1+\sqrt{d}}{2} = [\,a_0;\overline{a_1, a_2, \ldots, a_{\ell(\mathrm{d})-1}, 2a_0 - 1}\,]$ for $d \equiv 1(mod 4)$ where $\ell(\mathrm{d})$ is the period length in simple continued fraction expansion of algebraic integer w_d The fundamental unit ε_d of real quadratic number field is also denoted by$\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} > 1$ where $N\left(\varepsilon_d\right) = (-1)^{\ell(\mathrm{d})}$.

Determining the fundamental units of real quadratic fields is of great importance in the class number problem and the unit theorem since fundamental unit generates the unit group of real quadratic fields. Not all fundamental units are so easy to calculate practically, even for small values of d. So, this is very important to find a practical method so as to easily and rapidly determine fundamental unit ε_d.

The theorem of Friesen (1988) and Halter-Koch (1991) was examined a construction of infinite families of real quadratic fields with large fundamental units. Kawamoto and Tomita (2008) were determined minimal type of continued fraction for certain real quadratic fields. Also, Zhang and Yue (2014) proved some congruences about the coffecient of fundamental unit. Benamar et al. (2015), considered the real numbers which their quotient 's elements are a_i is the largest power of 2 dividing $i + 1$.

By using coefficients of fundamental unit, Yokoi (1990, 1991, 1993) defined two significant invariants as $m_d = \left[\!\left[\frac{u_d^2}{t_d}\right]\!\right]$ and $n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]$ where $[\![x]\!]$ represents the greatest integer not greater than x for class number problem and the solutions of Pell equation.

Lower bound of fundamental unit $\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} > 1$ of $k = \mathbb{Q}(\sqrt{d})$ was studied by Sasaki (1986) and Mollin (1996).

Originally, we examine the continued fraction expansions which have partial quotient elements repeated as 1s in the period length for w_d where d is a square free integer. Although there are infinitely many values of d having all 1s in the symmetric part of the period of continued fraction expansion of integral basis element, Tomita and Yamamuro (2002) described explicitly one type of d's in their paper for determining the fundamental units of the real quadratic fields.

We get infinitely many new real quadratic fields which have got the form of $w_d = \left[a_0; \overline{\underbrace{1,1,\ldots,1}_{\ell-1}, 2\,a_0} \right]$ or $w_d = \left[a_0; \overline{\underbrace{1,1,\ldots,1}_{\ell-1}, 2\,a_0 - 1} \right]$ where $d \equiv 2,3(mod4)$ or $d \equiv 1(mod4)$ respectively for the parametrization of square free positive integer d.

In this paper, we also classify the real quadratic fields in the main theorems according to arbitrary period length. For the classifications,we also determined the general forms of fundamental units ε_d and coefficents of fundamental units t_d, u_d in the terms of fibonacci sequence. Then, we get a fix on Yokoi's invariants and support all results with tables by the specialization. Also, we should say that the results obtained in this paper useful in the literature of class field theory.

2. Preliminaries

We need following definitions and lemmas which are useful for results in the next section.

Definition 2.1. $\{F_i\}$ is called as Fibonacci sequence if it is defined by the recurrence relation,

$$F_i = F_{i-1} + F_{i-2}$$

for $i \geq 2$, with seed values $F_0 = 0$ and $F_1 = 1$.

Note. For the set $I(d)$ of all quadratic irrational numbers in $\mathbb{Q}(\sqrt{d})$, we say that α in $I(d)$ is reduced if $\alpha > 1, -1 < \alpha' < 0$, ($\alpha'$ is the conjugate of α with respect to $\mathbb{Q}$), and denote by $R(d)$ the set of all reduced quadratic irrational numbers in $I(d)$.Then, it is well known that any number α in $R(d)$ is purely periodic in the continued fraction expansion and the denominator of its modular automorphism is equal to fundamental unit ε_d of $\mathbb{Q}(\sqrt{d})$.

Lemma 2.2. Let d be a square-free positive integer such that d congruent to 1 modulo 4. If we put $w_d = \frac{1+\sqrt{d}}{2}, a_0 = [w_d]$ into the $w_R = (a_0 - 1) + w_d$, then $w_d \notin R(d)$ but $w_R \in R(d)$holds. Moreover, for the period $l = \ell(\text{d})$ of w_R, we get $w_R = [\overline{2a_0 - 1, a_1, \ldots, a_{l-1}}]$and $w_d = [a_0, \overline{a_1, \ldots, a_{l-1}, 2a_0 - 1}]$.

Let $w_R = \frac{(P_l w_R + P_{l-1})}{(Q_l w_R + Q_{l-1})} = [\overline{2a_0 - 1, a_1, \ldots, a_{l-1}, w_R}]$ be a modular automorphism of w_R, then the fundamental unit ϵ_d of $Q(\sqrt{d})$ is given by the formulae

$$\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$$

$$t_d = (2a_0 - 1).\, Q_{\ell(d)} + 2Q_{\ell(d)-1}\,, \qquad u_d = Q_{\ell(d)}$$

where Q_i is determined by $Q_0 = 0, Q_1 = 1$ and $Q_{i+1} = a_iQ_i + Q_{i-1}$ $(i \geq 1)$.

Proof. Proof is in Tomita (1995).

Lemma 2.3. For a square free positive integer d congruent to 2,3 modulo 4, we put $w_d = \sqrt{d}$ and $a_0 = [\![\sqrt{d}]\!]$ into the $w_R = a_0 + w_d$. Then $w_d \notin R(d)$, but $w_R \in R\ (d)$ holds. Moreover, for the period $l = l(d)$ of w_R, we get

$$w_R = \left[\underbrace{\overline{2a_0,\ a_1, a_2, \ldots,\ a_{l(d)-1}}}_{\ell(d)}\right]$$

and

$$w_d = [a_0;\ \overline{a_1, a_2, \ldots,\ a_{l(d)-1}, 2a_0}\ ].$$

Furthermore, $w_R = \frac{w_R P_l + P_{l-1}}{w_R Q_l + Q_{l-1}} = [\ 2a_0, a_1, a_2, \ldots,\ a_{l(d)-1}, w_R]$ be a modular automorphism of w_R. Then the fundamental unit ε_d of $Q(\sqrt{d})$ is given by the following formula:

$$\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{\ell(d)} + Q_{\ell(d)-1}$$

$$t_d = 2a_0Q_{\ell(d)} + 2Q_{\ell(d)-1} \text{ and } u_d = 2Q_{\ell(d)}$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_iQ_i + Q_{i-1}$ $(i \geq 1)$.

Proof. It can be proved in a similar way of previous lemma.

Lemma 2.4. Let $d \equiv 2,3(mod4)$ be a square free positive integer and $a_0 = [\![\sqrt{d}]\!]$ denote the integer part of $w_d = \sqrt{d}$ for $d \equiv 2,3\ (mod4)$.If we consider w_d which includes partial constant elements repeated 1s in the case of period $l = l(d)$, then we have the continued fraction expansion as the form of

$$w_d = \sqrt{d} = [\, a_0; \overline{a_1,\, a_2, \dots,\, a_{\ell(\mathrm{d})-1},\, a_{\ell(\mathrm{d})}}\,] = [\, a_0; \overline{1,1, \dots ,1, 2a_0}\,]$$

$$w_R = a_0 + \sqrt{d} = [\, \overline{2a_0, 1,, \dots ,1}\,]$$

for integral basis element and reduced integral basis element respectively.

Furthermore, $A_k = a_0 F_{k+1} + F_k$ and $B_k = F_{k+1}$ are determined in the continued fraction expansion of w_d where $\{A_k\}$ and $\{B_k\}$ are two sequences defined by :

$$A_{-2} = 0 \quad , A_{-1} = 1 \quad , A_k = a_k A_{k-1} + A_{k-2}$$

$$B_{-2} = 1 \;,\; B_{-1} = 0 \quad ,\; B_k = a_k B + B_{k-2}$$

for $0 \leq k \leq \ell(d) - 1$ and

$$A_j = 2{a_0}^2 F_{\ell(\mathrm{d})} + 3a_0 F_{\ell(\mathrm{d})-1} + F_{\ell(\mathrm{d})-2}$$

$$B = 2a_0 F_{\ell(\mathrm{d})} + F_{\ell(\mathrm{d})-1}$$

for $k = \ell(\mathrm{d})$ where $l = \ell(\mathrm{d})$ is period length of $w_d = \sqrt{d}$ and $C_j = {A_j}/{B_j}$ is the j^{th} convergent in the continued fraction expansion of $\sqrt{d}$.

Moreover, $P_j = 2a_0 F_j + F_{j-1}$ and $Q_j = F_j$ are determined in the continued fraction $[b_1, b_2, b_3 \dots, b_\mathrm{n}, \dots] = [\, 2a_0, 1, 1, \dots ,1, \dots]$, where $\{P_j\}$ and $\{Q_j\}$ are two sequences defined by:

$$P_{-1} = 0 \quad , P_0 = 1, P_{j+1} = b_{j+1} P_j + P_{j-1} \text{ and}$$

$$Q_{-1} = 1 \;,\; Q_0 = 0,\; Q_{j+1} = b_{j+1} Q_j + Q_{j-1}$$

for $j \geq 0$.

Proof. We can prove by using mathematical induction. Using the following table which includes values of A_k, B_k and a_k, we can easily say that this is true for $k = 0$.

Table 1: (Converge of [$a_0; \overline{1,1,\dots,1,2a_0}$] for $l = l(d)$)

k	-2	-1	0	1	2	3	4	…
a_k			a_0	1	1	1	1	…
A_k	0	1	(a_0) $a_0F_1 + F_0$	(a_0+1) $a_0F_2 + F_1$	$(2a_0+1)$ $a_0F_3 + F_2$	$(3a_0+2)$ $a_0F_4 + F_3$	$(5a_0+3)$ $a_0F_5 + F_4$	…
B_k	1	0 F_0	1 F_1	1 F_2	2 F_3	3 F_4	5 F_5	…

Now, we assume that the result true for $k < i$ and $i \neq l.$ Using the defined the fibonacci sequence, we obtain ($a_i = 1$ for $1 \leq i \leq l-1$)

$$\begin{aligned} A_{k+1} = a_{k+1}A_k + A_{k-1} &= (a_0F_{k+1} + F_k) + (a_0F_k + F_{k-1}) \\ &= a_0(F_{k+1} + F_k) + (F_k + F_{k-1}) \\ &= a_0F_{k+2} + F_{k+1} \end{aligned}$$

$$B_{k+1} = a_{k+1}B_k + B_{k-1} = F_{k+1} + F_k = F_{k+2}$$

Moreover, since $a_l = 2a_0$ we get the following result :

$$A_l = 2a_0^2F_l + 3a_0F_{l-1} + F_{l-2}$$

$$B_l = 2a_0F_l + F_{l-1} \quad (\text{ for } k = l(d))$$

Furthermore, in the continued fraction $[b_1, b_2, b_3 \dots, b_n, \dots] = [\,2a_0, 1, 1, \dots, 1, \dots]$, we have following table and this completes the proof.

Table 2: (Converge of [$2a_0, 1, 1, \ldots, 1, \ldots$])

k	-1	0	1	2	3	4	…
b_k			$2a_0$	1	1	1	…
P_k	0	1	$(2a_0)$ $2a_0F_1 + F_0$	$(2a_0 + 1)$ $2a_0F_2 + F_1$	$(4a_0 + 1)$ $2a_0F_3 + F_2$	$(6a_0 + 2)$ $2a_0F_4 + F_3$	…
Q_k	1	0 F_0	1 F_1	1 F_2	2 F_3	3 F_4	…

Note: By defining similar new lemma, we can also obtain similar tables in the case of $w_d = \frac{1+\sqrt{d}}{2}$ for $d \equiv 1(mod4)$.

Remark 2.5. We can also write characteristic equation for Fibonacci sequence as the form of

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

for $k \geq 0$.

Remark 2.6. Let $\{F_n\}$ be Fibonacci sequence. Then, we state the following:

$$F_n \equiv \begin{cases} 0(mod4) & ; n \equiv 0(mod6) \\ 1(mod4) & ; n \equiv 1,2,5(mod6) \\ 2(mod4) & ; n \equiv 3(mod6) \\ 3(mod4) & ; n \equiv 4(mod6) \end{cases}$$

for n ≥ 0.

3. Main Theorems and Results

Main theorems and their results will be given with the notation of the preliminaries section as follows.

Theorem 3.1. (Main Theorem 1). Let d be square free positive integer and $\ell \geq 2$ be a positive integer satisfying that $3 \nmid \ell$. Suppose that the parametrization of d is

$$d = \left(\frac{(2\alpha+1)F_\ell + 1}{2}\right)^2 + (2\alpha+1)F_{\ell-1} + 1$$

where $\alpha \geq 0$ is a positive integer. Then following conditions hold:

(1) If $\ell \equiv 1(mod6)$ and α is even positive integer then $d \equiv 2(mod4)$ holds.
(2) If $\ell \equiv 2(mod6)$ and α is even positive integer then $d \equiv 3(mod4)$ holds.
(3) If $\ell \equiv 4(mod6)$ and α is even positive integer then $d \equiv 3(mod4)$ holds.
(4) If $\ell \equiv 5(mod6)$ and α is odd positive integer then $d \equiv 2(mod4)$ holds.

In the $\mathbb{Q}(\sqrt{d})$ real quadratic fields, we have

$$w_d = \left[\tfrac{(2\alpha+1)F_\ell+1}{2}; \overline{\underbrace{1,1,\dots,1}_{\ell-1}, (2\alpha+1)F_\ell + 1}\right] \text{ and } \ell = \ell(d)$$

for $d \equiv 2,3(mod4)$.

Additionally, we get the general form of fundamental unit ε_d and coefficients of fundamental unit t_d, u_d as follows:

$$\varepsilon_d = \left(\tfrac{(2\alpha+1)F_\ell+1}{2} + \sqrt{d}\right)F_\ell + F_{\ell-1},$$

$$t_d = (2\alpha+1)F_\ell^2 + F_\ell + 2F_{\ell-1} \quad \text{and} \quad u_d = 2F_\ell$$

Proof. It is clear that $d \notin Z_+$ for all $\ell \equiv 0\ (mod3)$ since Remark 2.6. We assume that $3 \nmid \ell$,$\ell \geq 2$ in order to get $d \in Z_+$. Originally, we have to show that four conditions hold as the followings:

(1) If $\ell \equiv 1\ (mod6)$, then $F_\ell \equiv 1(mod4)$ and $F_{\ell-1} \equiv 0(mod4)$ hold. If we consider that α is any even positive integer, then we obtain $d \equiv 2(mod4)$ by substituting these equivalences into the parametrization of d.

(2) If we consider $\ell \equiv 2(mod6)$, then we have $F_\ell \equiv 1(mod4)$ and $F_{\ell-1} \equiv 1(mod4)$. By substituting these values into parametrization of d and rearranging, we obtain $d \equiv 3(mod4)$ for any α even positive integer.
(3) If $\ell \equiv 4(mod6)$, then we have $F_\ell \equiv 3(mod4)$ and $F_{\ell-1} \equiv 2(mod4)$. By substituting these equivalences into the parametrization of d, we get $d \equiv 3(mod4)$ where α is any even positive integer.

(4) If $\ell \equiv 5(mod6)$ and α is odd positive integer then we get $F_\ell \equiv 1(mod4)$ and $F_{\ell-1} \equiv 3(mod4)$. So, we obtain $d \equiv 2(mod4)$.

Hence, conditions are satisfied.

By using Lemma 2.3 and Lemma 2.4, we get

$$w_R = \left(\frac{(2\alpha+1)F_\ell+1}{2}\right) + \left[\frac{(2\alpha+1)F_\ell+1}{2}; \overline{\underbrace{1,1,\dots,1}_{\ell-1}, (2\alpha+1)F_\ell+1}\right]$$

$$\Rightarrow \ w_R = \big((2\alpha+1)F_\ell+1\big) + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots + \cfrac{1}{1+\cfrac{1}{w_R}}}}}$$

$$= \big((2\alpha+1)F_\ell+1\big) + \frac{1}{1+}\cdots\frac{1}{+1+}\frac{1}{w_R}$$

Using Lemma 2.3 and the properties of continued fraction expansions, we have

$$w_R = \big((2\alpha+1)F_\ell+1\big) + \frac{F_{\ell-1}w_R + F_{\ell-2}}{F_\ell w_R + F_{\ell-1}}$$

If we rearrange the above equation, we obtain

$$w_R^2 - \big((2\alpha+1)F_\ell+1\big)w_R - \big(1+(2\alpha+1)F_{\ell-1}\big) = 0$$

This implies that $w_R = \left(\frac{(2\alpha+1)F_\ell+1}{2}\right) + \sqrt{d}$ since $w_R > 0$. If we consider Lemma 2.3 and Lemma 2.4, we get

$$\sqrt{d} = \left[\frac{(2\alpha+1)F_\ell+1}{2}; \overline{\underbrace{1,1,\dots,1}_{\ell-1}, (2\alpha+1)F_\ell+1}\right] \text{ and } \ell = \ell(d).$$

This shows that $w_d = \left[\frac{(2\alpha+1)F_\ell+1}{2}; \overline{\underbrace{1,1,\dots,1}_{\ell-1}, (2\alpha+1)F_\ell+1}\right]$ holds. So, the first part of proof is completed.

Now, we should determine ε_d, t_d and u_d using Lemma 2.3. We can get easily following equations:

$$Q_0 = 0 = F_0,\ Q_1 = 1 = F_1,\quad Q_2 = a_1.Q_1 + Q_0 \Rightarrow Q_2 = 1 = F_2$$
$$Q_3 = a_2 Q_2 + Q_1 = F_2 + F_1 = F_3,\quad Q_4 = F_4,\ \dots$$

So, this implies that $Q_i = F_i$ by using mathematical induction for $\forall i \geq 0$. If we substitute these values of sequence into the

$$\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = \left(a_0 + \sqrt{d}\right)Q_{\ell(d)} + Q_{\ell(d)-1}$$

and rearranged, we have

$$\varepsilon_d = \left(\frac{(2\alpha+1)F_\ell+1}{2} + \sqrt{d}\right)F_\ell + F_{\ell-1},$$

$$t_d = (2\alpha + 1)F_\ell^2 + F_\ell + 2F_{\ell-1} \quad \text{and} \quad u_d = 2F_\ell$$

which completes the proof of the Main Theorem 1.

Remark 3.2. Theorem 3.1 includes the parametrization of d depends on the arbitrary α positive integer parametre. The case that α=0 was tread in Tomita and Yamamuro (2002), but we should say that the present paper has got the most general theorem and results for such type of the real quadratic fields. Also, we can get infinitely many values of d which correspond to new $\mathbb{Q}\left(\sqrt{d}\right)$ for $\alpha \geq 1$ by using our results.

We obtain following results on Yokoi's invariants as well as fundamental unit and continued fraction expansion.

Corollary 3.3. Let d be square free positive integer and $\ell \geq 2$ be a positive integer satisfying that $\ell \equiv 5(mod6)$. Suppose that parametrization of d is

$$d = \left(\frac{1 + 3F_\ell}{2}\right)^2 + 3F_{\ell-1} + 1$$

then we have $d \equiv 2(mod4)$ and $w_d = \left[\frac{1+3F_\ell}{2}; \overline{\underbrace{1,1,\ldots,1}_{\ell-1}, 1 + 3F_\ell}\right]$ with $\ell = \ell(d)$.

Additionally, we get the fundamental unit ε_d, coefficients of fundamental unit t_d, u_d and Yokoi's invariant m_d as follows:

$$\varepsilon_d = \left(\frac{1+3F_\ell}{2} + \sqrt{d}\right)F_\ell + F_{\ell-1},$$

$$t_d = 3F_\ell^2 + F_\ell + 2F_{\ell-1} \quad \text{and} \quad u_d = 2F_\ell$$

$$m_d = 1$$

Besides, we state the following Table 3 where fundamental unit is ε_d, integral basis elemant is w_d and and Yokoi's invariant is m_d for $5 \leq \ell(d) \leq 23$.

Table 3: Square-free positive integers d with $5 \leq \ell(d) \leq 23$.

d	$\ell(d)$	m_d	w_d	ε_d
74	5	1	$[8; \overline{1,1,1,1,16}]$	$43+5\sqrt{74}$
18122	11	1	$[134; \overline{1,1,\dots,1,268}]$	$11981+89\sqrt{18122}$
5743778	17	1	$[2396; \overline{1,1,\dots,1,4792}]$	$3827399+1597\sqrt{5743778}$
1847849330	23	1	$[42986; \overline{1,1,\dots,1,85972}]$	$1231867513+28657\sqrt{1847849330}$

Proof. We create this result by subsituting $\alpha = 1$ into the Main Theorem 1. We have to determine value of m_d. We know that m_d is defined as $m_d = \left[\!\left[\frac{u_d^2}{t_d} \right]\!\right]$ in Yokoi's references. If we substitue t_d and u_d into the m_d, then we get

$$m_d = \left[\!\left[\frac{u_d^2}{t_d}\right]\!\right] = \left[\!\left[\frac{4F_\ell^2}{3F_\ell^2 + F_\ell + 2F_{\ell-1}}\right]\!\right]$$

Using assumption and the above equality, we get

$$2 > \left(\frac{4F_\ell^2}{3F_\ell^2 + F_\ell + 2F_{\ell-1}}\right) > 1{,}162$$

since Fibonacci sequence is monotone increasing. Therefore, we obtain $m_d = \left[\!\left[\frac{4F_\ell^2}{3F_\ell^2+F_\ell+2F_{\ell-1}}\right]\!\right] = 1$. This completes the proof of Corollary 3.3.

Also, Table 3 which can be extended for the different values of $\ell(d)$ is given numerical examples for Corollary 3.3.

Corollary 3.4. Let d be the square free positive integer and $\ell > 1$ be a positive integer such that $3 \nmid \ell, \ell \not\equiv 5(mod 6)$. We assume that the parametrization of d is

$$d = \left(\frac{1 + 5F_\ell}{2}\right)^2 + 5F_{\ell-1} + 1$$

then we get $d \equiv 2,3(mod 4)$ and $w_d = \left[\frac{1+5F_\ell}{2}; \overline{\underbrace{1,1,\dots,1}_{\ell-1}, 1 + 5F_\ell}\right]$ and $\ell = \ell(d)$.

Moreover, we have following equalities :

$$\varepsilon_d = \left(\left(\frac{1+5F_\ell}{2}\right)F_\ell + F_{\ell-1}\right) + F_\ell\sqrt{d}$$

$$t_d = 5F_\ell^2 + F_\ell + 2F_{\ell-1} \quad \text{and} \quad u_d = 2F_\ell$$

$$n_d = \begin{cases} 2 & ; if\ \ell = 2 \\ 1 & ; if\ \ell > 2 \end{cases}$$

for ε_d , t_d, u_d and Yokoi's invariant n_d.

Besides, we state the following Table 4 where fundamental unit is ε_d, integral basis elemant is w_d and Yokoi's invariant is n_d for $1 < \ell(d) \leq 13$. (In the following table, we rule out $\ell(d)$ = 4,8,10 since d is not a square free positive integer in these periods).

Table 4.: Square-free positive integers d with $2 \leq \ell(d) \leq 13$.

d	$\ell(d)$	n_d	w_d	ε_d
15	2	2	$[3; \overline{1,6}]$	$4+\sqrt{15}$
1130	7	1	$[33; \overline{1,1,\dots 1,66}]$	$437+13\sqrt{1130}$
340610	13	1	$[538; \overline{1,1,\dots 1,1166}]$	$135983+233\sqrt{340610}$

Proof. Corollary is obtained if we subsitute $\alpha = 2$ into the Main Theorem 1. So, we have to determine the value of Yokoi d-invariant n_d.

If we substitute t_d and u_d into the n_d and rearranged, then we obtain

$$n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right] = \left[\!\left[\frac{5F_\ell^2 + F_\ell + 2F_{\ell-1}}{4F_\ell^2}\right]\!\right]$$

If we put $\ell = 2$ above equation, we get $n_d = 2$. For $\ell > 2$, by using

$$n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right] = 1 + \left[\!\left[\frac{1}{4} + \frac{1}{4F_\ell} + \frac{F_{\ell-1}}{2F_\ell^2}\right]\!\right]$$

and considering that the Fibonacci sequence is monotone increasying, we obtain

$$0 < \frac{F_\ell^2 + F_\ell + 2F_{\ell-1}}{4F_\ell^2} < 0{,}293$$

Hence, we determine the value of Yokoi's invariant as $n_d = \left[\!\left[\frac{5F_\ell^2+F_\ell+2F_{\ell-1}}{4F_\ell^2}\right]\!\right] = 1$. Besides, Table 3.2 is given as an illustrate of this corollary.

Corollary 3.5. Let d be square free positive integer and $\ell \geq 2$ be a positive integer satisfying that $\ell \equiv 5(mod6)$. If we choose the parametrization of d as the form of

$$d = \left(\frac{7F_\ell + 1}{2}\right)^2 + 7F_{\ell-1} + 1$$

then $d \equiv 2(mod4)$ and $w_d = \left[\frac{7F_\ell+1}{2}; \overline{\underbrace{1,1,\dots 1}_{\ell-1}, 7F_\ell + 1}\right]$ with $\ell = \ell(d)$.

Moreover, we get the fundamental unit ε_d , coefficients of fundamental unit t_d, u_d and Yokoi's invariant n_d as follows:

$$\varepsilon_d = \left(\frac{7F_\ell+1}{2} + \sqrt{d}\right)F_\ell + F_{\ell-1},$$

$$t_d = 7F_\ell^2 + F_\ell + 2F_{\ell-1} \quad \text{and} \quad u_d = 2F_\ell$$

$$n_d = 1$$

Furthermore, we state the following Table 5 where fundamental unit is ε_d, integral basis elemant is w_d and and Yokoi's invariant is n_d for $5 \leq \ell(d) \leq 23$.

Table 5: Square-free positive integers d with $5 \leq \ell(d) \leq 23$.

d	$\ell(d)$	n_d	w_d	ε_d
346	5	1	$[18; \overline{1,1,1,36}]$	$93+5\sqrt{346}$
97730	11	1	$[312; \overline{1,1,\dots,1,624}]$	$27823+89\sqrt{97730}$
31255010	17	1	$[5590; \overline{1,\dots,1,11180}]$	$8928217+1597\sqrt{31255010}$
10060213978	23	1	$[100300; \overline{1,\dots 1,200600}]$	$2874314811 + 28657\sqrt{10060213978}$

Proof. This claim is obtained if we subsitute $\alpha = 3$ into the Main Theorem 1. It is sufficient to determine Yokoi's invariant value of $n_d = \left[\!\left[\frac{t_d}{u_d^2} \right]\!\right]$.

If we substitue t_d and u_d into the n_d, then we get

$$n_d = \left[\!\left[\frac{t_d}{u_d^2} \right]\!\right] = \left[\!\left[\frac{7F_\ell^2 + F_\ell + 2F_{\ell-1}}{4F_\ell^2} \right]\!\right]$$

Since $\{F_\ell\}$ is monotone increasing, we get following inequality

$$1 < \frac{7F_\ell^2 + F_\ell + 2F_{\ell-1}}{4F_\ell^2} \leq 1{,}86$$

for $\ell \geq 2$. Therefore, we obtain $n_d = \left[\!\left[\frac{7F_\ell^2 + F_\ell + 2F_{\ell-1}}{4F_\ell^2} \right]\!\right] = 1$ which completes the proof of Corollary 3.5. For the numerical examples, we give the Table 5.

Theorem 3.6. (Main Theorem 2) Let d be square free positive integer and $\ell \geq 2$ be a positive integer.

(1) We suppose that

$$d = (2\alpha F_\ell + 1)^2 + 8\alpha F_{\ell-1} + 4$$

where $\alpha > 0$ is a positive integer. In this case, we obtain that $d \equiv 1(\text{mod}4)$ and

$$w_d = \left[\alpha F_\ell + 1; \overline{\underbrace{1,1,\ldots,1}_{\ell-1}, 1 + 2\alpha F_\ell}\right]$$

with $\ell = \ell(d)$. Moreover, we get

$$t_d = 2\alpha F_\ell^2 + F_\ell + 2F_{\ell-1} \quad \text{and} \quad u_d = F_\ell$$

for $\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$.

(2) In the case that $\ell \equiv 0(mod3)$, if we assume that

$$d = (\alpha F_\ell + 1)^2 + 4\alpha F_{\ell-1} + 4$$

for $\alpha > 0$ positive odd integer, then $d \equiv 1(\text{mod}4)$ and

$$w_d = \left[1 + \frac{\alpha F_\ell}{2}; \overline{\underbrace{1,1,\ldots,1}_{\ell-1}, 1 + \alpha F_\ell}\right].$$

Also, in this case

$$t_d = \alpha F_\ell^2 + F_\ell + 2F_{\ell-1} \quad \text{and} \quad u_d = F_\ell$$

hold for $\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$.

Remark 3.7. it is clear that F_ℓ is odd number if $\ell \not\equiv 0(mod3)$ holds. $\frac{\alpha F_\ell}{2}$ is not integer if we substitue α odd integer into the papametrization of d fin the case of $\ell \not\equiv 0(mod3)$. So, we assume that ℓ is divided by 3 in the case of (2). Also, if we choose α is even integer, the parametrization of d coincides with the case of (1). That's why we assume $\ell \equiv 0(mod3)$ and $\alpha > 0$ positive odd integer in the case of (2).

Proof. (1) For any $\ell \geq 2$ and $\alpha > 0$ positive integer, $d \equiv 1(mod4)$ holds since $(2\alpha F_\ell + 1)$ is odd integer. From Lemma 2.2, we know that $w_d = \frac{1+\sqrt{d}}{2}, a_0 = [w_d]$ and $w_R = (a_0 - 1) + w_d$.

By using these equations, we obtain

$$w_R = \alpha F_\ell + \left[\alpha F_\ell + 1; \overline{\underbrace{1,1,\dots,1}_{\ell-1}, 1 + 2\alpha F_\ell}\right]$$

$$\Rightarrow \; w_R = (1 + 2\alpha F_\ell) + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots + \cfrac{1}{1 + \cfrac{1}{w_R}}}}}$$

$$= (1 + 2\alpha F_\ell) + \frac{1}{1} + \cdots + \frac{1}{1} + \frac{1}{w_R}$$

By a straight forward induction argument, we have

$$w_R = (1 + 2\alpha F_\ell) + \; + \frac{F_{\ell-1}w_R + F_{\ell-2}}{F_\ell w_R + F_{\ell-1}}$$

Using Definition 2.1 into the above equality, we obtain

$$w_R^2 - (1 + 2\alpha F_\ell)w_R - (1 + 2\alpha F_{\ell-1}) = 0$$

This implies that $w_R = \frac{(2\alpha F_\ell+1)+\sqrt{d}}{2}$ since $w_R > 0$. If we consider Lemma 2.2, we get $\frac{1+\sqrt{d}}{2} = \left[1+\alpha F_\ell; \overline{\underbrace{1,1,\dots,1}_{\ell-1}, 1+2\alpha F_\ell}\right]$ and $\ell = \ell(d)$. Proof of the first part of (1) is completed.

Now, we have to determine ε_d, t_d and u_d using Lemma 2.2. It is easily seen that $Q_i = F_i$ $(i \geq 0)$ by induction.

If we substitute the values of sequence into the coefficients of fundamental unit

$$t_d = 2\alpha F_\ell^2 + F_\ell + 2F_{\ell-1} \quad \text{and} \quad u_d = F_\ell$$

holds for $\varepsilon_d = \frac{t_d+u_d\sqrt{d}}{2}$.

(2) In the case of $\ell \equiv 0(mod3)$, we get $F_\ell \equiv 0(mod2)$. By subsituting this equivalence into the parametrization of d, we have $d \equiv 1(mod4)$ for $\alpha > 0$ positive odd integer.

By using Lemma 2.2 and the parametrization of $d = (\alpha F_\ell + 1)^2 + 4\alpha F_{\ell-1} + 4$, we have

$$w_R = (a_0 - 1) + w_d \Rightarrow \quad w_R = \frac{\alpha F_\ell}{2} + \left[1+\frac{\alpha F_\ell}{2}; \overline{\underbrace{1,1,\dots,1}_{\ell-1}, 1+\alpha F_\ell}\right]$$

$$\Rightarrow w_R = (1+\alpha F_\ell) + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots+\cfrac{1}{1+\cfrac{1}{w_R}}}}}$$

$$= (1+\alpha F_\ell) + \frac{1}{1} + \cdots + \frac{1}{1} + \frac{1}{w_R}$$

By a straight forward induction argument, we get

$$w_R = (1+\alpha F_\ell) + \frac{F_{\ell-1}w_R + F_{\ell-2}}{F_\ell w_R + F_{\ell-1}}$$

Using Definition 2.1 into the above equality, we obtain

$$w_R^2 - (1+\alpha F_\ell)w_R - (1+\alpha F_{\ell-1}) = 0$$

This implies that $w_R = \frac{\alpha F_\ell}{2} + \frac{1+\sqrt{d}}{2}$ since $w_R > 0$. If we consider Lemma 2.2, we get

$$\frac{1+\sqrt{d}}{2} = \left[1+\frac{\alpha F_\ell}{2}; \overline{\underbrace{1,1,\dots,1}_{\ell-1}, 1+\alpha F_\ell}\right] \text{ and } \ell = \ell(d).$$

Using $Q_i = F_i$ for $\forall i \geq 0$, we obtain the coefficients of fundamental unit

$$t_d = \alpha F_\ell^2 + F_\ell + 2F_{\ell-1} \quad \text{and} \quad u_d = F_\ell$$

for $\varepsilon_d = \frac{t_d+u_d\sqrt{d}}{2}$.

We can obtain following conclusions from Main Theorem 2.

Remark 3.8. We can say similar things of Remark 3.2 for Main Theorem 2.

Corollary 3.9. Let d be a square free positive integer congruent to 1 modulo 4. If we assume that d is satisfying the conditions in Main Theorem 2, then it always holds Yokoi's invariant m_d=0.

Proof. Yokoi's invariant m_d is defined $m_d = \left[\!\left[\frac{u_d^2}{t_d}\right]\!\right]$ by Yokoi (1991-1993). In the case of (1) if we substitue t_d and u_d into the m_d, then we obtain

$$m_d = \left[\!\left[\frac{u_d^2}{t_d}\right]\!\right] = \left[\!\left[\frac{F_\ell^2}{2\alpha F_\ell^2 + F_\ell + 2F_{\ell-1}}\right]\!\right]$$

So, we get m_d=0 since $t_d > u_d^2$ for $\alpha > 0$ positive integer.

In a similar way, we obtain m_d= $\left[\!\left[\frac{u_d^2}{t_d}\right]\!\right] = \left[\!\left[\frac{F_\ell^2}{\alpha F_\ell^2+F_\ell+2F_{\ell-1}}\right]\!\right]$ =0 with same reason ($t_d > u_d^2$ for $\alpha > 0$) in the case of (2).

Corollary 3.10. Let d be the square free positive integer corresponding to $\mathbb{Q}(\sqrt{d})$ holding (1) in the Main Theorem 2. We state the following Table 6 where fundamental unit is ε_d, integral basis element is w_d and Yokoi's invariant is n_d for $\alpha = 1,2$ and $2 \leq \ell(d) \leq 11$. (In this table, we rule out $\ell(d)$ = 6 for $\alpha = 1$ and $\ell(d)$ = 2 for $\alpha = 2$ since d is not a square free positive integer.)

Table 6: Square-free positive integers d with $2 \leq \ell(d) \leq 11$.

d	α	$\ell(d)$	n_d	w_d	ε_d
21	1	2	5	$[2;\overline{1,3}]$	$(5+\sqrt{21})/2$
37	1	3	3	$[3;\overline{1,1,5}]$	$(12+2\sqrt{37})/2$
69	1	4	2	$[4;\overline{1,1,7}]$	$(25+3\sqrt{69})/2$
149	1	5	2	$[6;\overline{1,\dots,1,11}]$	$(61+5\sqrt{149})/2$
797	1	7	2	$[14;\overline{1,\dots,1,27}]$	$(367+13\sqrt{797})/2$
1957	1	8	2	$[22;\overline{1,\dots,1,43}]$	$(929+21\sqrt{1957})/2$
4933	1	9	2	$[35;\overline{1,\dots,1,69}]$	$(2388+34\sqrt{4933})/2$
12597	1	10	2	$[56;\overline{1,\dots,1,111}]$	$(6173+55\sqrt{12597})/2$
32485	1	11	2	$[90;\overline{1,\dots,1,179}]$	$(16041+89\sqrt{32485})/2$
101	2	3	5	$[5;\overline{1,1,9}]$	$(20+2\sqrt{101})/2$
205	2	4	4	$[7;\overline{1,1,1,13}]$	$(43+3\sqrt{205})/2$
493	2	5	4	$[11;\overline{1,\dots,1,21}]$	$(111+5\sqrt{493})/2$
1173	2	6	4	$[17;\overline{1,\dots,1,33}]$	$(274+8\sqrt{1173})/2$
2941	2	7	4	$[27;\overline{1,\dots,1,53}]$	$(705+13\sqrt{2941})/2$

Table 6 (continued) : Square-free positive integers d with $2 \leq \ell(d) \leq 11$

d	α	$\ell(d)$	n_d	w_d	ε_d
7437	2	8	4	$[43; \overline{1, \ldots, 1, 85}]$	$\left(1811 + 21\sqrt{7437}\right)/2$
19109	2	9	4	$[69; \overline{1, \ldots, 1, 137}]$	$\left(4700 + 34\sqrt{19109}\right)/2$
49389	2	10	4	$[111; \overline{1, \ldots, 1, 221}]$	$\left(12223 + 55\sqrt{49389}\right)/2$
128333	2	11	4	$[179; \overline{1, \ldots, 1, 357}]$	$\left(31883 + 89\sqrt{128333}\right)/2$

Proof. This Corollary is obtained from main theorem by taking $\alpha = 1$ or 2 in the case of (1) of Main Theorem 2. We know n_d is defined $n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]$. If we substitue t_d and u_d into the n_d, then we get

$$n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right] = \left[\!\left[\frac{2\alpha F_\ell^2 + F_\ell + 2F_{\ell-1}}{F_\ell^2}\right]\!\right] = 2 + \left[\!\left[\frac{F_\ell + 2F_{\ell-1}}{F_\ell^2}\right]\!\right]$$

for $\alpha = 1$. For $\ell = 2$, we get $n_d = 5$ as well as $n_d = 4$ for $\ell = 3$. Since F_ℓ is monotone increasing sequence, we obtain

$$2{,}78 > \left(\frac{2\alpha F_\ell^2 + F_\ell + 2F_{\ell-1}}{F_\ell^2}\right) > 2$$

for $\ell \geq 4$. Also, in the case of $\alpha = 2$, we get $n_d = 5$ for $\ell = 3$ besides $n_d = 4$ for $\ell \geq 3$ by using similar way. The proof of Corollary 2 is completed.

Corollary 3.11. Let d be the square free positive integer positive integer corresponding to $\mathbb{Q}(\sqrt{d})$ holding (2) in the Main Theorem 2. We state the following Table 7 where fundamental unit is ε_d, integral basis element is w_d and Yokoi's invariant is n_d for $\alpha = 1,3$ and $3 \leq \ell(d) \leq 12$.

Table 7 : Square-free positive integers d with $3 \leq \ell(d) \leq 12$.

d	α	$\ell(d)$	n_d	w_d	ε_d
17	1	3	2	$[2;\overline{1,1,3}]$	$(8+2\sqrt{17})/2$
105	1	6	1	$[5;\overline{1,\dots,1,9}]$	$(82+8\sqrt{105})/2$
1313	1	9	1	$[18;\overline{1,\dots,1,35}]$	$(1232+34\sqrt{1313})/2$
21385	1	12	1	$[73;\overline{1,\dots,1,145}]$	$(21058+144\sqrt{21385})/2$
65	3	3	4	$[4;\overline{1,1,7}]$	$(16+2\sqrt{65})/2$
689	3	6	3	$[13;\overline{1,\dots 1,25}]$	$(210+8\sqrt{689})/2$
10865	3	9	3	$[52;\overline{1,\dots,1,103}]$	$(3544+34\sqrt{10865})/2$
188561	3	12	3	$[217;\overline{1,\dots,1,433}]$	$(62530+144\sqrt{188561})/2$

Proof. By subsituting $\alpha = 1$ or 3 into the (2) of Main Theorem 2, we get this corollary and the table in the case of (2) of Main Theorem 2. If we substitue t_d and u_d into the $n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]$. then we get

$$n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right] = \left[\!\left[\frac{\alpha F_\ell^2 + F_\ell + 2F_{\ell-1}}{F_\ell^2}\right]\!\right] = 1 + \left[\!\left[\frac{F_\ell + 2F_{\ell-1}}{F_\ell^2}\right]\!\right]$$

for $\alpha = 1$. We obtain $n_d = 2$ for $\ell = 3$ Since F_ℓ is increasing sequence, we obtain

$$1{,}282 > \left(\frac{\alpha F_\ell^2 + F_\ell + 2F_{\ell-1}}{F_\ell^2}\right) > 1$$

for $\ell > 3$. Also, if we take $\alpha = 3$, we get $n_d = 4$ for $\ell = 3$ as well as $n_d = 3$ for $\ell > 3$ in a similar way.

4. Conclusion

Quadratic fields have applications in different areas of mathematics such as quadratic forms, algebraic geometry, discrete mathematics, diophantine equations, algebraic number theory, computer science and even cryptography. In this paper, we are interested in the concept of real quadratic number fields. We considered the continued fraction expansions, fundamental unit and Yokoi invariants in the terms of Fibonacci sequence. Also, we established general important and interesting results for that. Results obtained in this paper provide us a practical method so as to rapidly determine continued fraction expansion of w_d, fundamental unit ε_d and Yokoi invariants n_d, m_d for period length $\ell(d)$. We are sure that these results help the researchers to enhance and promote their studies on quadratic fields to carry out a general framework for their applications in life.

References

Benamar, H., Chandoul, A. and Mkaouar, M. (2015). On the Continued Fraction Expansion of Fixed Period in Finite Fields. *Canad. Math. Bull.* **58**: 704-712.

Friesen, C. (1988). On continued fraction of given period. *Proc. Amer. Math. Soc.*, **103**: 9–14.

Halter-Koch, F. (1991). Continued fractions of given symmetric period. *Fibonacci Quart.*, **29**(4): 298–303.

Kawamoto, F. and Tomita, K. (2008). Continued fraction and certain real quadratic fields of minimal type. *J.Math.Soc. Japan*. **60**: 865–903.

Louboutin, S. (1988). Continued Fraction and Real Quadratic Fields. *J. Number Theory*. **30**: 167–176.

Mollin, R. A. (1996). *Quadratics.* Boca Rato, FL: CRC Press.

Olds, C. D. (1963). *Continued Functions*. New York: Random House.

Perron, O. (1950). *Die Lehre von den Kettenbrüchen.* New York, Chelsea, Reprint from Teubner Leipzig.

Sasaki, R. (1986). A characterization of certain real quadratic fields. *Proc. Japan Acad*. Ser. A. **62** (3): 97-100.

Sierpinski, W. (1964). *Elementary Theory of Numbers.* Warsaw: Monografi Matematyczne.

Tomita, K. (1995). Explicit representation of fundamental units of some quadratic fields. *Proc. Japan Acad*. Ser. A .**71** (2): 41-43.

Tomita, K. and Yamamuro, K. (2002). Lower bounds for fundamental units of real quadratic fields. *Nagoya Math. J.* **166**: 29-37.

Williams, K. S. and Buck, N. (1994). Comparison of the lengths of the continued fractions of $\sqrt{D}$ and $\frac{1}{2}(1+\sqrt{D})$. *Proc. Amer. Math. Soc*. **120**(4): 995-1002.

Yokoi, H. (1990). The fundamental unit and class number one problem of real quadratic fields with prime discriminant. *Nagoya Math. J.* **120**: 51-59.

Yokoi, H. (1991). The fundamental unit and bounds for class numbers of real quadratic fields. *Nagoya Math. J.* **124**: 181-197.

Yokoi, H. (1993). A note on class number one problem for real quadratic fields. *Proc. Japan Acad.* Ser. A. **69**: 22-26.

Yokoi, H. (1993). New invariants and class number problem in real quadratic fields. *Nagoya Math. J.* **132**: 175-197.

Zhang, Z. and Yue, Q. (2014). Fundamental units of real quadratic fields of odd class number. *Journal of Number Theory*. **137**: 122–129.